CALCULATION OF THE BOUNDARY LAYER ON AN ARBITRARY AXISYMMETRIC SURFACE ROTATING IN A STILL MEDIUM

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Boundary layer calculations on rotating surfaces make it possible to determine the friction and heat and mass transfer in the boundary layer. Problems of this type are encountered in turbine construction and other fields. The question has been thoroughly investigated for rotating disks and cylinders [1]. For surfaces of other configuration, only integral methods have been developed: by Howarth [2] and Nigam [3] for a sphere, and in [4] for an arbitrary surface. The necessity of satisfying two integral relations (instead of one, for two-dimensional flow) makes the calculations rather complex. It is advisable, therefore, to use for this purpose a class of exact similar solutions for the boundary layer on rotating surfaces, the existence of which has already been demonstrated by Geis [5].

The results of calculations on the basis of this class of similar solutions are given in this paper. An approximate method based on the use of similar solutions is developed for calculating the boundary layer on rotating surfaces of arbitrary shape.

§ 1. Transformation of equations. We examine the equations of the laminar boundary layer that forms on an axisymmetric surface rotating at a constant angular velocity ω in a medium at rest [1]:

$$u\frac{\partial u}{\partial x}-\frac{v^2}{r}\frac{dr}{dx}+w\frac{\partial u}{\partial z}=\nu\frac{\partial^2 u}{\partial z^2}$$
$$u\frac{\partial v}{\partial x}+\frac{uv}{r}\frac{dr}{dx}+w\frac{\partial v}{\partial z}=\nu\frac{\partial^2 v}{\partial z^2} \qquad (1.1)$$
$$\frac{\partial u}{\partial x}+\frac{u}{r}\frac{dr}{dx}+\frac{\partial w}{\partial z}=0.$$

Here are the coordinates (Fig. 1): x axis—along the intersection of the surface by a plane normal to the axis of rotation, and z axis—normal to the tangential plane; r(x) defines the meridional profile of the surface; u, v, ω, are the velocity-vector components that correspond to the x, y, z axes; and ν is the kinematic viscosity.

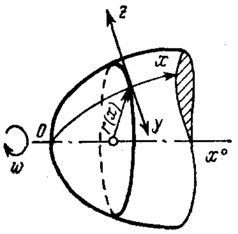

Fig. 1

Geis [5] has shown that similar solutions of the system (1.1) exist only if r(x) is a power

$$r = A\,(x+x_0)^m \qquad (1.2)$$

(or exponential) function of $x + x_0$. Indeed, let us introduce a stream function ψ such that

$$\partial\psi/\partial z = ru, \qquad \partial\psi/\partial x = -rw. \qquad (1.3)$$

We perform (for $m > 0$) the change of variable

$$\zeta = z\sqrt{\omega r'/\nu} \qquad (r' = dr/dx). \qquad (1.4)$$

We set

$$\psi = -{}^1/_2 r^2 H(\zeta)\sqrt{\nu\omega/r'}, \qquad u = r\omega F(\zeta), \qquad v = r\omega G(\zeta). \quad (1.5)$$

Then, if r(x) has the form (1.2), Eqs. (1.1) will transform into a system of ordinary differential equations

$$F'' = F^2 - G^2 + \beta HF', \qquad G'' = 2FG + \beta HG', \qquad H' + 2F = 0, \qquad (1.6)$$

where parameter β is expressed in terms of the exponent m in the form

$$\beta = \frac{1+3m}{4m}. \qquad (1.7)$$

For r decreasing with increasing x, i.e., for $m < 0$, $\dot r$ should be replaced by $-\dot r$ in the substitutions (1.4) and (1.5). Then, if F and H are exchanged for −F and −H, Eqs. (1.6) retain their form.

By simple computation, it is also possible to obtain an expression for

$$\frac{w}{\sqrt{\nu\omega r'}} = \beta H + 2(\beta - 1)F\zeta. \qquad (1.8)$$

We note that the case where r(x) is an exponential function of $(x + x_0)$ corresponds to $\beta = 3/4$.

The shape of the surface that corresponds to the exponential expression (1.2) depends on the constants A and m. In a rectangular system of coordinates (r, x^0), the shape of the generatrix is defined by the equations

$$x^\circ = \int_0^x \sqrt{1 - m^2A^2(x+x_0)^{2(m-1)}}\,dx, \qquad r = A\,(x+x_0)^m. \quad (1.9)$$

From $m < 0$, i.e., $\beta < 3/4$, the integrand has positive values starting from x_0, which corresponds to the initial radius r_0

$$x_0 = (mA)^{\frac{1}{1-m}}, \qquad r_0 = m^{\frac{m}{1-m}}A^{\frac{1}{1-m}}. \qquad (1.10)$$

For $0 < m \le 1$, i.e., $\beta \ge 1$, the surface commences with the minimum radius calculated from (1.10), the surface radius increasing from there on with increasing x. The case $m > 1$, i.e., $3/4 < \beta < 1$, corresponds to a surface that commences with the radius $r = 0$ and terminates with the radius r_0 as determined from (1.10).

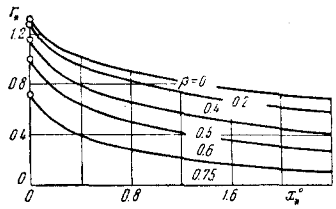

Fig. 2

From (1.9) it follows that all surfaces that correspond to a fixed value of m and variable values of A are similar, and the following equalities hold:

$$x_*^\circ = f(r_*), \qquad x_*^\circ = x^\circ : A^{\frac{1}{m-1}}, \qquad r_* = r : A^{\frac{1}{m-1}}. \qquad (1.11)$$

This explains why the dimensionless equations of motion (1.4) are independent of A. Figures 2 through 4 show the surface shapes calculated from Eqs. (1.9).

It should be noted that Eqs. (1.1) do not hold near the edge that correspond to the initial (or final) surface radius r_0. One should also remember that Eqs. (1.1) hold only in the case where the boundary-layer thickness is appreciably less than the corresponding value of r(x); this condition is no longer valid when β approaches 3/4, nor for small A.

The boundary conditions of the problem are

$$F(0) = H(0) = 0, \qquad G(0) = 1, \qquad F(\infty) = G(\infty) = 0 \qquad (1.12)$$

Note that for β = 1 we have the case of a rotating plane (with A = = 1) and also of a circular cone (with a cone angle of 2 arc sin A). A solution for this case was obtained by Cochran [6] and was later improved in connection with other problems [7, 8].

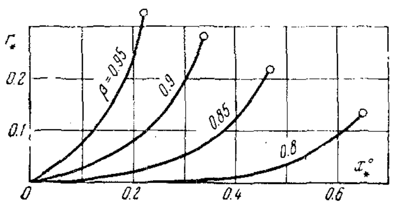

Fig. 3

§ 2. The case β = 0. For β = 0, a solution of the system (1.6) with the boundary conditions (6.12) can be obtained in closed form. Indeed, if the complex function

$$y = F + iG \qquad (2.1)$$

is introduced then, for β = 0, the system (1.6) reduces to the form

$$y'' = y^2, \qquad H' = -2\mathrm{Re}\,(y) \qquad (2.2)$$

while the boundary conditions (1.12) take the form

$$y(0) = i, \qquad y(\infty) = 0, \qquad H(0) = 0. \qquad (2.3)$$

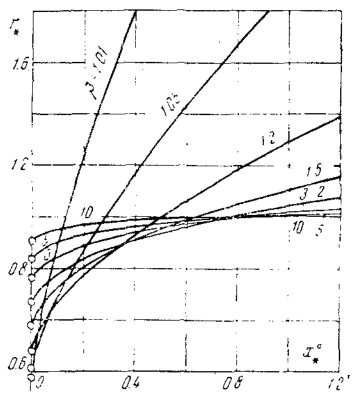

Fig. 4

Then

$$\frac{d(y')^2}{d\zeta} = 2y'y^2, \qquad (y')^2 = 2\int y^2 dy = \frac{2}{3}y^3 - c,$$

where, in accordance with (2.3), c = 0. Hence we have

$$d\zeta \,/\, dy = (^2/_3 y^3)^{-1/2}. \qquad (2.4)$$

Integrating (2.4) with allowance for (2.3), we get

$$y = 6 : [\zeta^2 + 2\sqrt{3}\zeta - i(2\sqrt{3}\zeta + 6)]. \qquad (2.5)$$

In this manner we obtain

$$F = 6\frac{\zeta^2 + 2\sqrt{3}\zeta}{(\zeta^2 + 2\sqrt{3}\zeta)^2 + (2\sqrt{3}\zeta + 6)^2}. \qquad (2.6)$$

$$G = 6\frac{2\sqrt{3}\zeta + 6}{(\zeta^2 + 2\sqrt{3}\zeta)^2 + (2\sqrt{3}\zeta + 6)^2}$$

$$H = -2\sqrt{6}\left[\frac{\sqrt{2}}{2} - \sqrt{3}\,\frac{\sqrt{\zeta^2 + 2\sqrt{3}\zeta + \sqrt{(\zeta^2 + 2\sqrt{3}\zeta)^2 + (2\sqrt{3}\zeta + 6)^2}}}{\sqrt{(\zeta^2 + 2\sqrt{3}\zeta)^2 + (2\sqrt{3}\zeta + 6)^2}}\right].$$

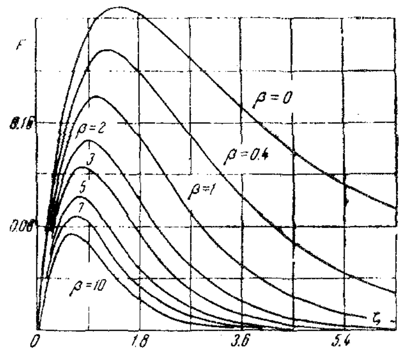

Fig. 5

Here,

$$F'(0) = -G'(0) = {}^1/_3\sqrt{3}, \qquad H(\infty) = -2\sqrt{3} \qquad (2.7)$$

Note that

$$\int_0^\infty G d\zeta = \mathrm{Im}\left(\int_0^\infty y d\zeta\right),$$

$$2\int_0^\infty G^2 d\zeta = \int_0^\infty (F^2 + G^2)\, d\zeta - \int_0^\infty (F^2 - G^2)\, d\zeta =$$

$$= \int_0^\infty y\bar{y} d\zeta + F'(0).$$

Taking (2.4) into account, for the boundary-layer characteristics we obtain the following thicknesses

$$B = \int_0^\infty G\, d\zeta = \sqrt{3}, \qquad C = \int_0^\infty G^2 d\zeta = \sqrt{3}\left(\frac{\pi}{4} - \frac{1}{3}\right). \qquad (2.8)$$

§ 3. Solution of the system of equations. The nonlinear boundary-value problem (1.6), (1.12) has been solved by the trial-and-error method, making use of interpolation.

Assigning approximate values of the missing boundary conditions in ζ = 0, $F'(0) = a$, $G'(0) = b$, we solve the problem with the initial conditions not only for these values but also for ($a + \Delta a$, b) and (a, b + + Δb), and then obtain by linear interpolation with respect to a and b, the improved corrections δa and δb to the initial a and b from the

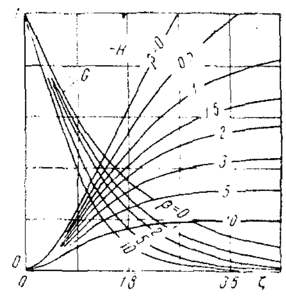

Fig. 6

condition $F(\infty) = G(\infty) = 0$. We then repeat the process, starting with $a + \delta a, b + \delta b$, and so forth.

Note that, due to the nonlinearity of the problem, the iteration process does not converge if the initial values of a and b are too roughly approximated.

The approximate values for a and b are determined by interpolation, first from known values for $\beta = 0$ (§ 2) and $\beta = 1$ [6], and then on the basis of solutions obtained for other values of β. The difficulty associated with an infinite limit of integration is overcome by taking into account that, starting with a certain finite value of $\zeta = \zeta^*$ ($\zeta^* = 12$ for $\beta \geq 1$), the unknown functions already have their values almost at infinity, specifically $F(\zeta^*) = 0$, $G(\zeta^*) = 0$. Having performed the calculation for a sufficiently large value of ζ^* – of the order of 12 – we extend the calculations to a still larger value of ζ^*. If there is no change in the results, the process of increasing the accuracy of the solution is terminated at this point.

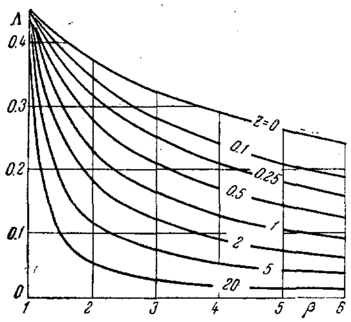

Fig. 7

System (1.6) is integrated by Merson's modification of the Runge-Kutta method [9]. The accuracy of the calculations was up to $\varepsilon = 10^{-7}$ for each step, the boundary conditions being satisfied with the same accuracy.

Some results of calculations performed for $\beta = 1$ are given below; results of other authors are given for comparison:

$a = F'(0)$	$b = G'(0)$	$-H(\infty)$	B	C
0.510233	–0.615922	0.88447	1.27144	0.672527
[8] 0.510233	–0.615922	0.88446	—	—
[7] 0.510	–0.6159	0.8845	1.271	0.6721
[6] 0.510	–0.616	0.886	—	—

It can be seen that there is good correlation between the data. For verification purposes we can use the relation

$$\int_0^\infty FG\,d\zeta = -G'(0) : 2(1+\beta)$$

which follows from (1.6). Calculations show that the relation is satisfied with an accuracy of 10^{-7}. The principal results of the calculations are given in a table and in Figs. 5 and 6. The solution was obtained on a Ural-2 computer, programmed by A. Z. Serazetdinov.

§ 4. Approximate method for an arbitrary rotating surface. We will use a set of solutions for various values of β to develop an approximate method for calculating the boundary layer on a rotating surface of arbitrary shape. For this purpose, the given surface is broken down in separate regions, each of which is approximated by a surface governed by the power law $r = A(x + x_0)^m$. As a basis for the calculations, we take the change in boundary layer thickness

$$\Delta_y = \int_0^\infty \left(\frac{v}{r\omega}\right)^2 dz = C(\beta)\left(\frac{\nu}{\omega r}\right)^{1/2}. \tag{4.1}$$

Let x_1 and x_2 be the beginning and end, respectively, of one of the regions of the surface. The boundary-layer parameters are known for x_1 and have to be determined for x_2. Let a surface from the family (1.2) pass through x_1 and x_2. Then

$$\frac{r_2}{r_1} = \left(\frac{x_2 + x_0}{x_1 + x_0}\right)^m = \left(1 + \frac{x_2 - x_1}{x_1 + x_0}\right)^m. \tag{4.2}$$

Since $r = mr/(x + x_0)$, from (4.1) we have

$$1 / (x_1 + x_0) = C^2\nu / \omega m r_1 \Delta_{y1}^2. \tag{4.3}$$

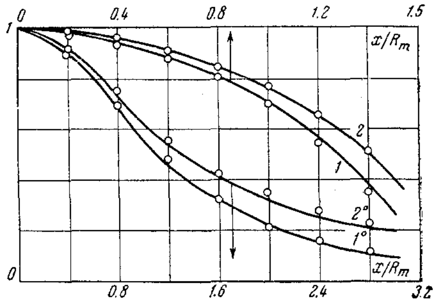

Fig. 8

β	$F'(0)$	$-G'(0)$	$-H(\infty)$	B	C
0	0.577350	0.577350	3.46410	1.73205	0.782999
0.15	.566170	.583070	2.26063	.59623	.760727
.30	.555675	.588801	1.66504	.50527	.741181
.45	.544934	.594552	1.35307	.43314	.723858
.60	.534671	.600364	1.16548	.37826	.708160
.75	.524996	.606141	1.03755	.33288	.693829
.85	.518889	.610039	0.969198	.30650	.684947
.95	.513052	.613958	.910708	.28259	.676553
1.0	.510233	.615922	.884475	.27144	.672527
.1	.504783	.619850	.837017	.25052	.664792
.2	.499576	.623771	.795184	.23124	.657453
.3	.494598	.627677	.757981	.21338	.650477
.4	.489834	.631561	.724642	.19676	.643837
.5	.485272	.635417	.694562	.18123	.637507
2.0	.465073	.654174	.579152	.11620	.609756
3	.434162	.688635	.443089	.02530	.567934
4	.411243	.719243	.363984	0.962485	.537274
5	.393277	.746662	.311545	.915179	.513388
6	.378632	.771498	.273912	.877614	.493996
7	.366352	.794224	.245425	.846681	.477778
8	.355829	.815201	.223014	.820534	.463910
9	.346659	.834704	.204862	.797985	.451843
10	.338558	.852950	.189819	.778229	.441196

Substituting this expression into (4.2) and denoting

$$Z = \frac{x_2 - x_1}{r_1} \frac{\nu}{\omega \Delta_{y_1}^2}$$
$$\Lambda = \frac{r_2 - r_1}{x_2 - x_1} \left(\Delta_{y_1}^2 \frac{\omega}{\nu} \right) \qquad (4.4)$$

we obtain the formula

$$\Lambda = [(1 + ZC^2/m)^m - 1] : Z \,. \qquad (4.5)$$

From the known values of C(β), it is possible to construct, with the aid of this formula, a series of curves Λ(β) for various values of Z (Fig. 7). By analogy with formula (4.3), we may write

$$\left(\Delta_y^2 \frac{\omega}{\nu} \right)_2 = \frac{C^2}{mr_2} (x_2 + x_0)$$

and then eliminate x_0 as in (4.3). Finally, we get

$$\left(\Delta_y^2 \frac{\omega}{\nu} \right)_2 = \left(\Delta_y^2 \frac{\omega}{\nu} \right)_1 \frac{r_1}{r_2} \left(1 + ZC^{2/m} \right). \qquad (4.6)$$

Having determined from the values x_1, x_2, r_1, r_2, and the dimensionless boundary layer thickness $(\Delta_y^2 \omega/\nu)$ at point x_1 the values of Z and Λ from (4.4), we find from Fig. 7 the corresponding value of β that is in one-to-one correspondence with the number C^2/m. It is then possible from (4.6) to determine the boundary-layer thickness at the point x_2, and so forth. Thus, passing from point to point, it is possible to determine from the β values obtained, not only Δ_y, but also all the other boundary-layer characteristics, in particular, the friction stress components

$$\frac{\tau_x}{\rho (r\omega)^2} = F'(0) \left(\frac{\nu r'}{r^2 \omega} \right)^{1/2}, \qquad \frac{\tau_y}{\rho (r\omega)^2} = G'(0) \left(\frac{\nu r'}{r^2 \omega} \right)^{1/2}, \qquad (4.7)$$

the displacement thickness

$$\delta_y^* = \int_0^\infty G \, d\zeta \left(\frac{\nu}{\omega r'} \right)^{1/2} \qquad (4.8)$$

and the velocity profiles.

It should be remembered that the method proposed is applicable to surfaces for which $\dot{r} \neq 0$.

Note that an analogous method was previously proposed by Smith [10] for two-dimensional flows.

We will compare our results with those obtained by other methods. Howarth [2], Nigam [3], and the author [4] have calculated the boundary layer on a rotating sphere by the method of integral relations. It has been shown that Nigam's assumption of a constant boundary-layer thickness is not justified as confirmed by experimental evidence obtained by Kobashi [11]. Nor does Nigam's result hold concerning the "break" in the boundary layer at large distances from the equator. Recent experiments by Bowden and Lord [12] showed that such a "break" does not exist; from the equator there is emitted a thin radial jet created by the collision of fluid masses flowing in from the two hemispheres.

Let us calculate the ratios of the local values of the skin-friction stress components at the variable radius r to the corresponding values for a disk (denoted by the superscript°) at the same radius and the same ω, ρ, and ν. The values obtained by our method (points in Fig. 8) correlate well with the results obtained in [4] by an integral method (continuous lines) both for τ_x/τ_x° (curve 1) and τ_y/τ_y° (curve 2). It should be noted, however, that the dimensionless friction stress components (4.7) themselves are determined more accurately by our method.

An analogous calculation was performed for a half-body of revolution whose shape derives from the superposition of a uniform flow on the flow from a three-dimensional source. This result also agrees well with calculation by the method of integral relations [4] (curves 1° and 2°).

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